UNIQUENESS OF THE SOLUTION OF THE INVERSE PROBLEM FOR THE COEFFICIENTS OF THE LINEAR HEAT-CONDUCTION EQUATION

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We consider the problems of unambiguous determination of coefficients of the linear heat-conduction equation from measurements on a temperature field.

In recent years, studies of complex phenomena and the interpretation of experimental results have been based on the solution of inverse problems of mathematical physics. An important role in the theoretical and practical design is therefore played by the problem of unique determination of the required quantities from observations.

In the inverse problem for the coefficients, a part of the problem lies in the properties of the transformation of the observed image onto the set of coefficients of the original equation. It was shown in [1-4] that this transformation is not unique for the linear and nonlinear heat-conduction equation. It was then shown that nonunique thermophysical media can exist which correspond to a single temperature field, and the functional properties of these fields were found. However, several important questions remained unanswered. Here belong the nature of ambiguity in mapping the temperatures onto the set of thermophysical constants, conditions for which this ambiguity occurs, and also the properties of the differential equations which lead to these cases. The present work is directed towards filling this gap, and is devoted to developing a method for the investigation of one-to-one correspondence between the coefficients, and a solution of a differential equation. We shall always assume that the direct problem is correctly formulated which ensures the existence of a unique temperature field which depends continuously on the starting data.

We make the following definitions.

Definition 1. The solution of an equation is called unidentifiable as a whole if the one-to-one correspondence between this solution and the coefficients of an equation which it satisfies is violated.

We shall now single out problems for which the one-to-one correspondence is not satisfied by any solutions.

Definition 2. The formulation of a boundary-value problem is called unidentifiable as a whole if each function which satisfies this problem is unidentifiable as a whole on the full set of its coefficients.

It is also necessary to note that the required quantities may not be unique.

Definition 3. A set of coefficients which corresponds to a single solution of the boundary-value problem is called the subset of ambiguity.

These definitions separate the solutions of differential equations that correspond to nonunique coefficients of the equation. Consequently, the solutions of inverse problems which are generated by the formulation of such direct problems cannot have a unique solution, disregarding the uniqueness of the solution of the direct problem.

We shall use this viewpoint to consider the properties of smooth solutions of the inhomogeneous linear heat-conduction equation which can be differentiated a sufficient number of times. Let us suppose that in the region $Q_T = \{(x, t): 0 < x < 1, 0 < t < T\}$:

$$a_1 \frac{\partial u}{\partial t} = a_2 \frac{\partial^2 u}{\partial x^2} + f(x, t), \quad (x, t) \in Q_T,$$
$$u|_{t=0} = \varphi(x), \quad x \in (0, 1), \tag{1}$$

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$$\alpha_{01}u|_{x=0} - \alpha_{02}\frac{\partial u}{\partial x}\Big|_{x=0} = \gamma_0(t), \quad t\in(0,\ T),$$

$$\alpha_{11}u|_{x=1} + \alpha_{12}\frac{\partial u}{\partial x}\Big|_{x=1} = \gamma_1(t), \quad t\in(0,\ T),$$

where $f(x, t) \in C^{2,1}(\bar{Q}_T)$; $\varphi(x) \in C^2[0, 1]$; $\gamma_{0,1}(t) \in C^1[0, T]$ are known functions; α_{01}, α_{02}, α_{11}, α_{12} are known coefficients; $a_{1,2}$ = constant are coefficients whose one-to-one correspondence to the solution of the problem (1) forms the problem in hand. It is shown in [5] that the specification of smooth starting data ensures the existence of a unique differentiable solution $u(x, t) \in C^{2,1}(\bar{Q}_T)$.

THEOREM. The solution of the problem (1) which is unidentifiable as a whole has the form

$$u^* = \rho^{-1}\int_0^t f^*(x,\ \tau)\,d\tau + \varphi(x), \quad (x,\ t)\in\bar{Q}_T. \tag{2}$$

For its existence, it is necessary and sufficient that one can specify a unique family

$$a_1 - \lambda a_2 = \rho, \tag{3}$$

the boundary conditions are consistent:

$$\begin{aligned} \alpha_{01}\varphi|_{x=0} - \alpha_{02}\frac{d\varphi}{dx}\Big|_{x=0} &= \gamma_0|_{t=0}, \\ \alpha_{11}\varphi|_{x=1} + \alpha_{12}\frac{d\varphi}{dx}\Big|_{x=1} &= \gamma_1|_{t=0}, \end{aligned} \tag{4}$$

and that the function f^* satisfies the following conditions:

$$\begin{aligned} \lambda\frac{\partial f^*}{\partial t} &= \frac{\partial^2 f^*}{\partial x^2}, \quad (x,\ t)\in Q_T, \\ \lambda f^*|_{t=0} &= \rho\frac{d^2\varphi}{dx^2}, \quad x\in(0,\ 1), \\ \alpha_{01}f^*|_{x=0} - \alpha_{02}\frac{\partial f^*}{\partial x}\Big|_{x=0} &= \rho\frac{d\gamma_0}{dt}, \quad t\in(0,\ T), \\ \alpha_{11}f^*|_{x=1} + \alpha_{12}\frac{\partial f^*}{\partial x}\Big|_{x=1} &= \rho\frac{d\gamma_1}{dt}, \quad t\in(0,\ T), \end{aligned} \tag{5}$$

where λ, p = const.

Proof of the Theorem. Necessity. We assume that two vectors $a' \neq a''$, where $a = \{a_1, a_2\}$, correspond to a single solution u^*. We can then obtain

$$(a_1' - a_1'')\frac{\partial u^*}{\partial t} = (a_2' - a_2'')\frac{\partial^2 u^*}{\partial x^2}, \quad (x,\ t)\in Q_T. \tag{6}$$

Condition (6) implies a linear dependence of the differential terms of the problem (1), i.e.,

$$\lambda\frac{\partial u^*}{\partial t} = \frac{\partial^2 u^*}{\partial x^2}, \quad (x,\ t)\in Q_T, \tag{7}$$

where $\lambda = (a_1' - a_1'')/(a_2' - a_2'') \neq 0$. Condition (7) reduces the original heat-conduction equation to the form

$$\frac{\partial u^*}{\partial t} = \rho^{-1}f, \quad (x,\ t)\in Q_T, \tag{8}$$

where the parameter ρ is determined by formula (3), which expresses the character of the required subset of ambiguity A^*. The solution of the problem (1) is then determined by the quadrature (2).

Comparing (8) with the result of application of the operator $\partial^2/\partial x^2$ on the solution (2), we obtain

$$\lambda f = \int_0^t \frac{\partial^2 f}{\partial x^2} d\tau + \rho \frac{d^2\varphi}{dx^2}, \quad (x, t) \in Q_T,$$

Hence it follows

$$\lambda \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}, \quad (x, t) \in Q_T, \tag{9}$$

and also

$$\lambda f|_{t=0} = \rho \frac{d^2\varphi}{dx^2}, \quad x \in (0, 1). \tag{10}$$

Acting consecutively by the boundary operators of the problem (1) on function (2), we obtain the following equations:

$$\begin{aligned} &\alpha_{01} f|_{x=0} - \alpha_{02} \frac{\partial f}{\partial x}\bigg|_{x=0} = \rho \frac{d\gamma_0}{dt}, \quad t \in (0, T), \\ &\alpha_{11} f|_{x=1} + \alpha_{12} \frac{\partial f}{\partial x}\bigg|_{x=1} = \rho \frac{d\gamma_1}{dt}, \quad t \in (0, T), \end{aligned} \tag{11}$$

and a correspondence of the form (4).

Thus, the assumption of existence of different coefficients which have a single solution implies the necessary validity of conditions (3)-(5).

Sufficiency. We consider the function

$$w = \lambda \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \in C^{2,1}(Q_T),$$

whose existence and differential properties follow from the differentiability of the solution $u(x, t)$ of the problem (1), and of the function $f(x, t)$. By acting by the operators $\partial/\partial t$ and $\partial^2/\partial x^2$ on the original heat-conduction equation and comparing the obtained results according to condition (9), we obtain the following homogeneous equation:

$$a_1 \frac{\partial w}{\partial t} = a_2 \frac{\partial^2 w}{\partial x^2}, \quad (x, t) \in Q_T.$$

If, from the set of coefficients of the problem (1), one chooses those that belong to the family (3), we find $w = [\lambda f - \rho(\partial^2 u/\partial x^2)]/a_1$ and $w = [\rho(\partial u/\partial t) - f]/a_2$. Using (3), (4), (9), and (10), these two expressions make it possible to obtain the homogeneous boundary conditions:

$$\begin{aligned} &w|_{t=0} = 0, \quad x \in (0, 1), \\ &\alpha_{01} w|_{x=0} - \alpha_{02} \frac{\partial w}{\partial x}\bigg|_{x=0} = 0, \quad t \in (0, T), \\ &\alpha_{11} w|_{x=1} + \alpha_{12} \frac{\partial w}{\partial x}\bigg|_{x=1} = 0, \quad t \in (0, T). \end{aligned}$$

In this case, it follows from the principle of the maximum that the function $w \equiv 0$ vanishes. Consequently, if the conditions of the Theorem are satisfied, the solution of the problem (1) has linearly dependent terms (7).

We now consider the functional

$$J(a) = \iint\limits_{Q_T} \left(a_1 \frac{\partial u}{\partial t} - a_2 \frac{\partial^2 u}{\partial x^2} - f\right)^2 dxdt.$$

The given function u(x, t) is the solution of the problem (1) if the coefficients $a_{1,2}$ satisfy the system

$$a_1 \iint\limits_{Q_T} \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} dxdt - a_2 \iint\limits_{Q_T} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2} dxdt = \iint\limits_{Q_T} f \frac{\partial u}{\partial t} dxdt,$$

$$a_1 \iint\limits_{Q_T} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2} dxdt - a_2 \iint\limits_{Q_T} \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial x^2} dxdt = \iint\limits_{Q_T} f \frac{\partial^2 u}{\partial x^2} dxdt,$$

which expresses the minimum of the functional $J(a)$ with respect to the variation of $a_{1,2}$. The linear dependence (7) of the terms of the heat-conduction equation then indicates that the principal determinant and minors of the system are equal to zero. Hence follows the existence of nonunique values $a_{1,2}$ for which the given function u(x, t) satisfies the problem (1). Function (2) is a solution if the boundary condition (1) of the problem (1) is consistent according to expression (4).

Thus, the conditions of Theorem (3)-(5) imply the violation of the one-to-one correspondence.

We now prove the uniqueness of the family (3). The inhomogenous terms in (9)-(11) determine uniquely the parameters of the family λ and ρ. Consider the indeterminacy implied by these conditions. If $d\gamma_{0,1}/dt = 0$ and

$$\alpha_{01} f|_{x=0} - \alpha_{02} \left.\frac{\partial f}{\partial x}\right|_{x=0} = 0, \quad t \in (0, T),$$

$$\alpha_{11} f|_{x=1} + \alpha_{12} \left.\frac{\partial f}{\partial x}\right|_{x=1} = 0, \quad t \in (0, T),$$

then conditions (11) are satisfied for arbitrary ρ. In this case, it follows from (10) and (9) that if $d^2\varphi/dx^2 = 0$, the violation of the identifiability takes place if $f \equiv 0$. Then, the nontriviality of quadrature (2) implies $\rho = 0$, and the arbitrariness of the parameter λ indicates that the thermophysical properties are independent of external conditions. In the more general case when $d^2\varphi/dx^2 \neq 0$ the quantity ρ is determined by condition (10), where the parameter λ is found from (9).

If $\partial f/\partial t = 0$ and $\partial^2 f/dx^2 = 0$, then, using the solution (2), the heat-conduction equation gives the parameters

$$\rho = \frac{a_1 f|_{t=0}}{f|_{t=0} - a_2 \dfrac{d^2\varphi}{dx^2}}, \qquad \lambda = \rho \frac{\dfrac{d^2\varphi}{dx^2}}{f|_{t=0}}, \quad x \in (0, 1), \tag{12}$$

which exist and are unique for all $a_{1,2}$ except for the value $a_2^* = f|_{t=0}/(d^2\varphi/dx^2)$. The assignment of this coefficient corresponds to the case of stationary solution of the problem (1) when the initial state remains unchanged and $u \equiv \varphi(x)$.

For $f|_{t=0} = 0$ and $d^2\varphi/dx^2 = 0$, the parameters of the family are determined by conditions (9) and (11). In the opposite case, it follows from (2) that the violation of identifiability is associated with $f \equiv 0$.

If $f|_{t=0} = 0$ and

$$\alpha_{01} f|_{x=0} - \alpha_{02} \frac{\partial f}{\partial x}\bigg|_{x=0} = 0, \quad t \in (0, T),$$

$$\alpha_{11} f|_{x=1} + \alpha_{12} \frac{\partial f}{\partial x}\bigg|_{x=1} = 0, \quad t \in (0, T),$$

for any $d\gamma_{0,1}/dt$ and $d^2\varphi/dx^2$ (including zero) for $f \neq 0$, condition (9) ensures the identifiability of the heat-conduction equation.

Thus, in the case of nonzero boundary values of the function f and of the derivatives $d\gamma_{0,1}/dt$ and $d^2\varphi/dx^2$, there also exists a unique family (3). This completes the proof.

This theorem completely answers the questions formulated above. First, the nature of violation of the one-to-one correspondence between the solution and coefficients of the direct problem is expressed by a family from the subset of ambiguity A^* which generates the subspace of solutions U^* which are unidentifiable as a whole. Second, the necessary condition for the existence of $u^* \in U^*$ is associated with specifying a definite form of external interaction which corresponds to consistent boundary functions. Third, nonidentifiability as a whole occurs if and only if the differential terms of the equation are linearly dependent.

For example, the problem

$$a_1 \frac{\partial u}{\partial t} = a_2 \frac{\partial^2 u}{\partial x^2} + 1,$$

$$u|_{t=0} = x(x-1), \quad u|_{x=0} = t, \quad u|_{x=1} = t$$

has the subset of ambiguity $A^* = \{a : a_1 - 2a_2 = 1\}$ which corresponds to the solution $u^* = t + x(x-1)$. It is then impossible, in inverse problems which use observations of the temperature field u^*, to find unique values of the coefficients $a_{1,2}$.

From this viewpoint, the form of the solution (2) which is unidentifiable as a whole, and conditions (4) and (5), can be used as a basis for the simultaneous determination of several coefficients of the equation from data about its unique solution $u \notin U^*$. The violation of only one of them ensures a one-to-one relationship between the vector of the required coefficients, and the functions which satisfy the direct problem.

We shall analyze the obtained results. Using (5), one can find a unique function f^* if the boundary conditions are consistent, but the converse is not true. In the six equations (4) and (5), there are eight parameters which characterize their behavior and coupling. A given function f^* therefore corresponds to a two-parameter family of consistent boundary conditions. In this case, the initial values of the functions $\varphi(x)$ and $\gamma_{0,1}(t)$ act as the parameters.

It follows from (5) that the nontrivial value of the function f^*, i.e., $f \equiv 0$, is excluded if we have homogeneous boundary conditions. This property is associated with the following

COROLLARY 1. If, in the problem (1), the function $f \not\equiv 0$, its subset of ambiguity is empty for linear initial temperature distribution and constant boundary conditions, or for the following homogeneous properties of the external interaction:

$$f|_{t=0} = 0, \quad x \in (0, 1),$$

$$\alpha_{01} f|_{x=0} - \alpha_{02} \frac{\partial f}{\partial x}\bigg|_{x=0} = 0, \quad t \in (0, T),$$

$$\alpha_{11} f|_{x=1} + \alpha_{12} \frac{\partial f}{\partial x}\bigg|_{x=1} = 0, \quad t \in (0, T).$$

This case excludes the values $d^2\varphi/dx^2 = 0$ and $d\gamma_{0,1}/dt = 0$ which also determine

COROLLARY 2. If $f \equiv \text{const} \neq 0$, the problem (1) is unidentifiable as a whole for constant boundary conditions of the second kind, $\alpha_{01} = 0$, $\alpha_{11} = 0$, $\gamma_{0,1} \equiv \text{const}$ and an initial distribution of the form

$$\varphi(x)=\frac{1}{2}\left(\frac{\gamma_1}{\alpha_{12}}-\frac{\gamma_0}{\alpha_{02}}\right)x^2+\frac{\gamma_0}{\alpha_{02}}x+\text{const},\quad \gamma_0\alpha_{12}\neq\gamma_1\alpha_{02}.$$

This result shows that the subset of ambiguity of a boundary-value problem can contain an infinite number of families of coefficients of the equation under study. Then, each solution of such direct problems can be found using quadrature (2). In this case, conditions (5) also determine uniquely the parameters λ and ρ. We note that the direct problems which are generated by the formulation of such direct problems cannot have a unique solution at all.

An example of a formulation which is unidentifiable as a whole is the problem

$$a_1\frac{\partial u}{\partial t}=a_2\frac{\partial^2 u}{\partial x^2}+f,$$

$$u|_{t=0}=x(x-1),\quad -\frac{\partial u}{\partial x}\bigg|_{x=0}=1,\quad -\frac{\partial u}{\partial x}\bigg|_{x=1}=-1,$$

whose solutions for $f\equiv$ constant $\neq 0$ are described by the quadrature (2).

It is easily seen that the model (1) is also unidentifiable as a whole if $f\equiv 0$. The subset of ambiguity then consists of families of the form $\lambda = a_1/a_2$. The arbitrariness in the choice of the parameter λ indicates that this ratio is independent of the boundary conditions, and the parameter $\rho = 0$ shows that the conditions for existence of such family are satisfied.

If one chooses the values of parameters λ and ρ from (12),

$$f^*=a_2\left(\frac{\gamma_0}{\alpha_{02}}-\frac{\gamma_1}{\alpha_{12}}\right), \tag{13}$$

the problem, determined by Corollary 2, then has a stationary solution which coincides with the initial temperature distribution $u\equiv\varphi$. This result makes it possible to obtain

COROLLARY 3. If the conditions of Corollary 2 are satisfied, the thermophysical system described by model (1) cannot be obtained from the initial state by external interaction of the type (13).

We investigate the violation of the one-to-one correspondence from the viewpoint of other known properties of differential equations. The existence of the subset of ambiguity A* leads to the invariance of the solution of the direct problem with respect to the family of coefficients which belong to A*. The solution of the heat-conduction equation in the region Q_T is invariant with respect to a transformation of coefficients if and only if the following heat flux is incident on the boundary $\partial Q_T = \{(x, t):\ x = 0,\ x = 1,\ 0 < t < T\}$:

$$q^*|_{\partial Q_T}=-\frac{a_2}{\rho}\int_0^t\frac{\partial f}{\partial x}\bigg|_{\partial Q_T}d\tau-a_2\frac{d\varphi}{dx}\bigg|_{\partial Q_T},\quad t\in(0,\ T).$$

The solution (2) is then invariant with respect to the transformation $a_1'' = a_1' + c$, $a_2'' = a_2' + c/\lambda$, which represents the translation group acting on the set of coefficients of the heat-conduction equation.

We shall show that the invariance with respect to a coefficient transformation of a solution of a differential equation forms an invariance group generated by stretching the coordinate system. We consider, in the region $G = \{(x, y) : 0 < x < 1,\ 0 < y < 1\}$, the following equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,\ y),\quad (x,\ y)\in G.$$

By stretching the coordinate system $x' = k_1x$, $y' = k_2y$ we then obtain

$$k_1^2\frac{\partial^2 u}{\partial x'^2}+k_2^2\frac{\partial^2 u}{\partial y'^2}=f(x',\ y'),\quad (x',\ y')\in G',$$

where $G' = \{(x', y') : 0 < x' < k_1,\ 0 < y' < k_2\}$. The last equation has linearly independent differential terms $\lambda(\partial^2 u/\partial x'^2) = \partial^2 u/\partial y'^2$, $\lambda \neq 0$, which are associated with the invariance of the solution $u(x', y')$ with respect to the family of the stretching coefficients $k_1^2 + \lambda k_2^2 = \rho$, where the parameters λ and ρ can be found from arguments analogous to those above.

Summarizing the obtained results, we can make the following conclusions. In the formulation of inverse problems for the coefficients one should bear in mind the possibility of violation of the one-to-one correspondence between the solution and coefficients of the equation under study. The violation of the one-to-one correspondence is associated with the existence of a subset of the required quantities which determines the solution of the inverse problem with accuracy up to some family. This subset corresponds to a certain definite solution of the direct problem which is invariant with respect to the family of coefficients from the subset of ambiguity. Inhomogeneous formulations of correctly formulated problems are possible, all of whose solution display the invariance of coefficients. The conditions for violation of the one-to-one correspondence are associated with an invariance group of stretching the coordinate system.

In conclusion, we note that the present study develops a method directed towards the principal possibility of finding simultaneously several parameters of boundary-value problems, and the obtained results throw a new light on the properties of the heat-conduction equation.

NOTATION

x, spatial coordinate; t, time; a_1, specific heat; a_2, thermal conductivity; α_{01}, α_{02}, α_{11}, α_{12}, coefficients of the boundary conditions; $u(x, t)$, temperature field; $f(x, t)$, strength of the volume heat sources; $\varphi(x)$, initial temperature distribution; $\gamma_{0,1}(t)$, boundary functions; $C^{2,1}$, C^2, C^1, classes of differentiable functions; $u^*(x, t)$, unidentifiable solution; U^*, subspace of unidentifiable solution; A^*, subset of ambiguity — λ, ρ, its parameters; Q_T, G, regions of variation of the independent variables; and $k_{1,2}$, stretching coefficients of the coordinate system.

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